Team Round Solutions

LMT Fall 2024

December 14, 2024

1. **[20]** A positive integer n is called "foursic" if there exists a placement of 0 in the digits of n such that the resulting number a multiple of 4. For example, 14 is foursic because 104 is a multiple of 4. Find the number of two-digit foursic numbers.

   Proposed by: Benjamin Yin

   Solution. $\boxed{45}$

   All even numbers are foursic (place a 0 at the end). If we insert a digit in the ones digit of an odd number it will not be divisible by 4, and otherwise it won't even be even, so it can't be divisible by 4. Therefore, the answer is $\boxed{45}$. $\square$

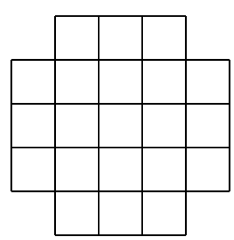
2. **[25]** Currently, Selena's analog clock says $4:00$. Suddenly her clock breaks, so the hour hand moves 12 times as fast as it normally does, but the minute hand stays the same speed. Find the degree measure of the smaller angle formed by the minute and the hour hand 2024 minutes from now.

   Proposed by: Muztaba Syed

   Solution. $\boxed{120}$

   The minute hand normally moves 12 times as fast as the hour hand, so they are now at the same speed. The angle between them will never change, remaining at $\boxed{120}$. $\square$

3. **[25]** Jason starts in a cell of the grid below. Every second he moves to an adjacent cell (i.e., two cells that share a side) that he has not visited yet. Find the maximum possible number of cells that Jason can visit.

   Proposed by: Muztaba Syed

   Solution. $\boxed{19}$

   Checkerboard the grid. Initially there are 13 black and 12 white cells, and after removing corners there are 9 black and 12 white cells. Every time Jason moves the color of his cell changes, so he can visit at most $9 \cdot 2 + 1 = \boxed{19}$ cells. This can easily be constructed. $\square$

4. **[30]** A rhombus has vertices at $(0,0)$, $(6,8)$, $(16,8)$, and $(10,0)$. A line with slope m passes through the point $(3,1)$ and splits the rhombus into 2 regions of equal area. Find m.

   Proposed by: Muztaba Syed

Solution. $\boxed{\dfrac{3}{5}}$

This line has to pass through the center of the rhombus, which has x coordinate 8 and y coordinate 4, meaning the slope is $\frac{4-1}{8-3} = \boxed{\dfrac{3}{5}}$. □

5. **[30]** There are distinct quadratics $e(x)$, $p(x)$, $h(x)$, $r(x)$, $a(x)$, and $m(x)$ with leading coefficient 1, such that their roots are 2 distinct values from the set $\{3,4,5,6\}$. James takes three of these quadratics, sums two, and subtracts the last. Given that this new quadratic has a root at 0, find its other root.

   Proposed by: Edwin Zhao

Solution. $\boxed{5}$

Suppose the polynomials are $e(x)$, $p(x)$, and $h(x)$. Looking at the constant terms we see $e(0) + p(0) - h(0) = 0$, and the possible constant terms are 12, 15, 18, 20, 24, and 30. It is easy to see this forces the constants to be 12, 18, and 30. Thus the polynomial is

$$(x-3)(x-4)+(x-3)(x-6)-(x-5)(x-6) = x^2-5x.$$

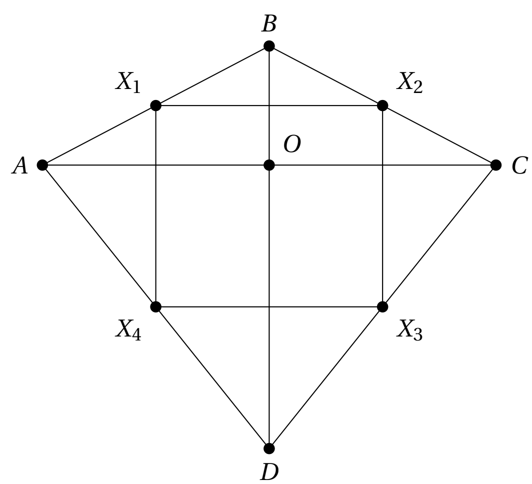
So the other root is $x = \boxed{5}$. □

6. **[35]** A kite with $AB = BC$ and $AD = CD$ has diagonals which satisfy $AC = 80$ and $BD = 71$. Let AC and BD intersect at a point O. Find the area of the quadrilateral formed by the circumcenters of ABO, BCO, CDO, and ADO.

   Proposed by: Calvin Garces

Solution. $\boxed{1420}$

Let the circumcenters of ABO, BCO, CDO, ADO be X_1, X_2, X_3, and X_4, respectively.



Because AC is perpendicular to BD, and $AO = OC$, we see that $X_1X_2 = X_3X_4 = 40$. Similarly, $X_2X_3 = X_4X_1 = \frac{71}{2}$. Also notice that $\angle X_1X_2X_3 = 90°$ so $X_1X_2X_3X_4$ is a rectangle. The area is $40 \cdot \frac{71}{2} = \boxed{1420}$. □

7. **[35]** Let A, F, L, M, and T be distinct digits such that $\overline{FALL} + \overline{LMT} = 2024$ and $F, L > 0$. Find the sum of all possible values of $\overline{FAT}$.

   Proposed by: Adam Ge

Solution. $\boxed{542}$

Notice that $F = 1$ and then casework on the value of L. We see that $1024 = \overline{AMT} + \overline{LLL}$ by rearranging the digits. So we can simply subtract multiples of 111 from 1024 to get all solutions.

The possible tuples (F, A, L, M, T) are $(1,0,9,2,5)$, $(1,3,6,5,8)$, $(1,4,5,6,9)$, and $(1,5,4,8,0)$ which yields an answer of $\boxed{542}$. □

8. **[40]** Let a and b be positive integers such that $10 < \gcd(a,b) < 20$ and $220 < \operatorname{lcm}(a,b) < 230$. Find the difference between the smallest and largest possible values of ab.

   Proposed by: Muztaba Syed

   Solution. $\boxed{1596}$

   Recall that $\gcd(a,b) \mid \operatorname{lcm}(a,b)$ and $ab = \gcd(a,b)\cdot\operatorname{lcm}(a,b)$. Notice that the value of the gcd has a more powerful impact on the size of ab.

   The maximum value is achieved when $\gcd(a,b) = 19$ and $\operatorname{lcm}(a,b) = 228$. For the minimum we see there are no multiples of 11 between 220 and 230, but 228 is a multiple of 12. So the answer is $228\cdot 19 - 228\cdot 12 = \boxed{1596}$. □

9. **[45]** Five friends named Ella, Jacob, Muztaba, Peter, and William are suspicious of their friends for having secret group chats. Call a group of three people a "secret chat" if there is a chat with just the three of them (there cannot be multiple chats with the same three people). They have the following perfectly logical conversation in this order:

   - Ella: I am part of 5 secret chats.
   - Jacob: I know all of the secret chats that Ella is in.
   - Muztaba: Peter is in all but one of my secret chats.
   - Peter: I am in a secret chat that William cannot know exists.
   - William: I share exactly two secret chats with Jacob and two secret chats with Peter.

   Let E be the number of chats Ella is in, J the number of chats Jacob is in, M the number of chats Muztaba is in, P the number of chats Peter is in, and W the number of chats William is in. Find $10000E + 1000J + 100M + 10P + W$.

   Proposed by: Muztaba Syed

   Solution. $\boxed{54354}$

   Each person can be part of at most $\binom{4}{2} = 6$ chats and Ella is in 5. If Jacob knows which chat Ella could be in that does not exist, it must have both of them (i.e., exactly one of EJM, EJP, EJW does not exist). We also know EMP, EMW, EPW must exist.

   Muztaba tells us that there is no chat he is in that Peter is not in. The chat EJM would violate this, meaning EJM does not exist which implies that EJP and EJW exist.

   We can also rule out JMW since Muztaba is in it and Peter is not. At this point we know EMP, EMW, EPW, EJP, EJW exist; EJM, JMW do not exist; and are unsure about JMP, JPW, and MPW. William knows this information, so Peter's dialogue tells us that JMP exists, since it is the only chat William does not know about.

   Finally we need to figure out JPW and MPW. Without these William is in exactly one chat with Jacob and one chat with Peter (EJW and EPW). This means that JPW exists and MPW does not.

   In total the chats that exist are EMP, EMW, EPW, EJP, EJW, JMP, JPW. The chats that don't are EJM, JMW, and MPW. We can count to see Ella is in 5, Jacob is in 4, Muztaba is in 3, Peter is in 5, and William is in 4. So the answer is $\boxed{54354}$. □

10. **[45]** Find the sum of all positive integers $n \le 2024$ such that all pairs of distinct positive integers (a,b) that satisfy $ab = n$ have a sum that is a perfect square.

    Proposed by: Benjamin Yin

    Solution. $\boxed{326}$

    Notice that all n that satisfy the conditions must have $n+1 = m^2$ for some positive integer m. Next, we can rewrite n as $m^2 - 1 = (m-1)(m+1)$. Thus, the sum $(m-1)+(m+1) = 2m$ must also be a perfect square. Thus, we can rewrite m as $m = 2r^2$ for another integer r. Now, we can rewrite n as $n = 4r^4 - 1$. Testing $r = 1, 2, 3, 4$ gives two solutions: $n = 3, 323$. Thus, the final answer is $3 + 323 = \boxed{326}$. □

11. **[50]** Let $\phi = \frac{1+\sqrt{5}}{2}$. Find

    $$\left(4+\phi^{\frac{1}{2}}\right)\left(4-\phi^{\frac{1}{2}}\right)\left(4+i\phi^{-\frac{1}{2}}\right)\left(4-i\phi^{-\frac{1}{2}}\right).$$

    Proposed by: Evin Liang

Solution. $\boxed{239}$

Let $f(x) = x^2 - x - 1$. Then $f(x)$ has roots ϕ and ϕ^{-1}. Thus $f(x^2)$ has roots $\phi^{\frac{1}{2}}$, $-\phi^{\frac{1}{2}}$, $i\phi^{-\frac{1}{2}}$, and $-i\phi^{-\frac{1}{2}}$. So we have

$$\left(4+\phi^{\frac{1}{2}}\right)\left(4-\phi^{\frac{1}{2}}\right)\left(4+i\phi^{-\frac{1}{2}}\right)\left(4-i\phi^{-\frac{1}{2}}\right) = f(4^2) = \boxed{239}.$$

□

12. **[50]** Eddie assigns each of Jason, Jerry, and Jonathan a different positive integer. The three are each perfectly logical and currently know that their numbers are distinct but don't know each other's numbers. Additionally, if one of them knows the answer to the question they will say so immediately. They have the following conversation listed below in chronological order:

- Eddie: Does anyone know who has the smallest number?
- Jason, Jerry, Jonathan (at the same time): I'm not sure.
- Jonathan: Now I know who has the smallest number.
- Eddie: Does anyone know who has the largest number?
- Jason, Jonathan, Jerry (at the same time): I'm not sure.
- Jerry: Now I know who has the largest number.
- Jason: Wow, our numbers are in an geometric sequence!

Find the sum of their numbers.

Proposed by: Samuel Tsui

Solution. $\boxed{14}$

From the first bit of information Jonathan must have 2 because if he had 1 he would say that first and if he had a larger number he wouldn't be sure. Then the second bit of information implies Jerry has 4 by a similar argument. Thus, Jason has 8 giving the answer of $\boxed{14}$. □

13. **[55]** 2 identical red tokens and 2 identical black tokens are placed on distinct cells of a 5×5 grid. Suppose it is impossible to color some additional cells of the grid red or black such that there exists a red path between the red tokens and a black path between the black tokens. Find the number of possible arrangements of the tokens on the grid.

(A red path is a path of edge adjacent red cells, and same for a black path.)

Proposed by: Chris Cheng, Samuel Tsui, Edwin Zhao

Solution. $\boxed{3736}$

We proceed with casework.

- All 4 tokens are on on outer cells of the grid, and they are alternating in color. there are $25-9=16$ squares on the outside of the grid, so there are $\binom{16}{4}$ ways to choose 4 of these squares and 2 ways to arrange the colors of the tokens, so this yields $\binom{16}{4}\cdot 2 = 3640$ cases.
- The tokens are arranged in a 2×2 square, in a checkerboard pattern so that the colors are alternating. There are 16 ways to choose the location of this square, and 2 ways to arrange the colors of the tokens, so this yields $16\cdot 2 = 32$ cases.
- One token is in a corner and two other tokens of the opposite are to each side of it, preventing traveling anywhere from there. There are 4 ways to choose the corner, 2 ways to choose the colors, and 8 ways to choose where the last token should be, as the other 14 places it can go all result in overcounting. This yields $4\cdot 2\cdot 8 = 64$ cases.

Thus, our final answer is $3640+32+64 = \boxed{3736}$ possibilities. □

14. **[55]** Let $ABCD$ be an isosceles trapezoid with $2DA = 2AB = 2BC = CD$. A point P lies in the interior of $ABCD$ such that $BP = 1$, $CP = 2$, $DP = 4$. Find the area of $ABCD$.

Proposed by: Muztaba Syed

Solution. $\boxed{\dfrac{21\sqrt{3}}{4}}$

Note that $\triangle PBC \sim \triangle PCD$ with ratio 2. Since $\angle DCB = 60^\circ$, this means $\angle BPC = 120^\circ$. By Law of Cosines this means $BC = \sqrt{7}$. The area is then $3 \cdot \frac{7\sqrt{3}}{4} = \boxed{\dfrac{21\sqrt{3}}{4}}$ □

15. **[60]** Amy has a six-sided die which always rolls values greater than or equal to the previous roll. She rolls the die repeatedly until she rolls a 6. Find the expected value of the sum of all distinct values she has rolled when she finishes.

Proposed by: Muztaba Syed

Solution. $\boxed{\dfrac{223}{20}}$

Consider the face numbered k. Consider the first time she rolls a value from k to 6. If it is any of the values $k+1$, $k+2$, ..., 6, then she will never roll k. But each of these outcomes is equally likely, so the probability that k is rolled is $\frac{1}{7-k}$. Thus the expected value is

$$\frac{1}{6}+\frac{2}{5}+\frac{3}{4}+\frac{4}{3}+\frac{5}{2}+\frac{6}{1}=\boxed{\frac{223}{20}}.$$

□